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**Abstract:** In this paper, a non parametric identifier for homogeneous nonlinear systems affine in the input is proposed. The identification algorithm is based on the neural networks using sigmoidal activation functions. The learning algorithm is derived by means of Lyapunov function method and homogeneity theory. A numerical example demonstrates the performance of the proposed identifier.

*Keywords:* Differential Neural Network, Nonlinear Systems, Homogeneous systems, Identification.

## 1. INTRODUCTION

Analysis and design of dynamic control systems need a valid mathematical model of the plant. However, most control systems have model uncertainties. Therefore, the tools looking for a valid mathematical description of the dynamic system are widely demanded and applied.

Nonlinear system identification is a field of control theory (Ljung, 2006), which develops algorithms of mathematical modeling of control systems based on input and output signals measured on-line or/and during some experiments. Identification problem has been tackled using different approaches due to a large class of system models and its inherent complexity. The identification process in general needs to handle the available input and output data in order to postulate a model and validate it somehow. Some of the most popular identification techniques are: functional series methods, frequency domain approaches, fuzzy models and neural networks (NNs) (see e.g. Billings (1980), Haber and Keviczky (1999), Nelles (2013) and Haykin (1994) for more details).

NNs are a special kind of approximation algorithms having the theoretical capability to approximate a large class of nonlinear mappings, (see Hornik et al. (1989)). Differential neural networks (DNN) are utilized for approximation of dynamical systems (Poznyak et al., 2001), (Lewis et al., 1998), (Sontag, 1993), since they can be trained on-line (in a real time). DNNs can process many inputs and outputs, so they are applicable to multi-variable systems.

The DNN identification admits the selection of different activation functions, which represent a certain basis for the model of the system in an admissible space, for example, sigmoidal functions, polynomials or radial basis functions (Cybenko, 1989), (Diaconis and Shahshahani, 1984), (Hubbert, 2002). The adjustment of the time-varying parameters (weights) in the DNN according its structure and the set of activation functions should be adjusted, for example, by a stability analysis based on Lyapunov procedure (see e.g. Chairez (2017), Jagannathan and Lewis (1996)). In this work the DNN identification algorithms are developed for a specific class of systems: homogeneous nonlinear systems affine in the input.

Homogeneity is a symmetry-like property under which an object remains consistent with respect to a certain scaling or dilation. Homogeneous systems can be utilized for local approximations (Hermes, 1986), (Andrieu et al., 2008) or set-valued extensions (Orlov, 2005), (Levant, 2005) of nonlinear control systems. In particular, some models of process control (Zimenko et al., 2017), nonholonomic mechanical systems (Pomet and Samson, 1993) and systems with frictions (Orlov, 2005) are homogeneous or at least locally homogeneous.

To the best of authors knowledge, identification problem of homogeneous systems is not well studied in the literature. One of the main features of homogeneous system is that an analysis of its behavior in a whole state space can be reduced to a similar analysis on a unit sphere (Hermes, 1995), (Bernuau et al., 2014), (Polyakov et al., 2016). This feature implies a specific structure of the DNN identifier in this work. The activation functions are selected to approximate the systems on the unit sphere, next due to homogeneity the system model can be expanded to the whole space.

This paper is organized as follows: In Section 2, the class of uncertain nonlinear systems considered in the identification scheme are described. In Section 3, the approximation property

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of NNs for homogeneous systems is studied. Identification algorithms as well as their convergence are discussed in Section 4. Section 5 presents the numerical results in order to demonstrate the performance of the identifiers. Section 6 concludes this manuscript with some remarks.

*Notation:* $\mathbb{R}_+ = \{x \in \mathbb{R} \colon x \geq 0\}$, where $\mathbb{R}$ denotes the set of real numbers. For any $\vartheta \in \mathbb{R}_+$ and $\forall x \in \mathbb{R}$ we set $\lceil x \rfloor^{\vartheta} = \mathrm{sign}(x)|x|^{\vartheta}$. The Euclidean norm is denoted by $\|.\|$. For $M \in \mathbb{R}^{m\times n}$, $M = [m_1, m_2, \ldots, m_n]$, we set $\mathrm{vec}(M) = [m_1^\top, m_2^\top, \ldots, m_n^\top]^\top$. The Kronecker product is denoted by $\otimes$.

## 2. PROBLEM STATEMENT

Let us consider a nonlinear affine control system:

$$\dot{x} = f(x,u) := f_0(x) + \sum_{i=1}^{m} f_i(x) u_i, \tag{1}$$

- $x \in \mathbb{R}^n$ is the state vector of the system;
- $u = (u_1, .., u_m)^\top \in \mathbb{R}^m$ is the control input, $m \leq n$;
- $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, 1, ..., m$ are unknown nonlinear vector fields;

The system is studied under the following basic assumptions:

*Assumption 1. The vector fields $f_i, i = 0, 1, ..., m$ are continuous on the unit sphere:*

$$S = \{x \in \mathbb{R}^n : \|x\| = 1\}. \tag{2}$$

*Assumption 2. The vector fields $f_i$ are homogeneous in the standard sense with known degrees $\nu_i \in \mathbb{R}_+$, i.e.*

$$f_i(\lambda x) = \lambda^{\nu_i} f_i(x), \quad \forall x \in \mathbb{R}^n,\ \forall \lambda > 0, \tag{3}$$

*where $i = 0, 1, ..., m$.*

The standard homogeneity means that the function $f$ is symmetric with respect to dilation $x \mapsto \lambda x$ of its first argument. The generalized concepts of homogeneity have been developed for other types of dilation $D : \mathbb{R} \to \mathbb{R}^{n\times n}$, $x \mapsto D(\lambda)x$, and for both finite and infinite dimensional systems, (see e.g. Zubov (1958), Khomenuk (1961), Kawski (1995), Polyakov et al. (2016)). The matrix $D(\lambda)$ denotes a dilation in $\mathbb{R}^n$. In this manuscript we consider the standard homogeneity property since, in the finite dimensional case, any generalized homogeneous system is topologically equivalent to a standard homogeneous one (Grüne, 2000), (Polyakov, 2018).

*Assumption 3. It is assumed that the whole state vector of* (1) *is bounded and sufficiently excited, i.e.*

$$0 < \|x(t)\| < +\infty, \quad \forall t \geq 0$$

*and it can be measured, i.e. $h(x) = x$.*

The **main goal** of the identification problem for the control system (1) under assumptions 2 and 3 is to find the model representation for the vector field $f$ and the parameters for that model, such that, the error between the states of the real system and the proposed structure is bounded and small enough. In other words, the first step is to represent (1) with a valid model for its approximation. In this work we use a DNN structure for this purpose. Then, the second step is to use a DNN identifier structure and design the adaptive laws for the adjustment weights, such that, the error

$$e := x - \hat{x} \tag{4}$$

between the system states and the identifier state $\hat{x}$ tends to zero or, at least, it is bounded, i.e. $\limsup_{t\to\infty} \|e\| \leq \beta < +\infty$.

Homogeneity allows local properties of vector fields to be extended globally. For example (see e.g. Bhat and Bernstein (2005) and Polyakov (2018)), a vector field $f_i : \mathbb{R}^n \to \mathbb{R}^n$ satisfying Assumption 2, is Lipschitz continuous on $\mathbb{R}^n \setminus \{0\}$ if and only if it satisfies Lipschitz condition on the unit sphere (2). Similarly, the system (1) satisfying the Assumptions 1 and 2 has the right-hand side continuous (on the first argument) in $\mathbb{R}^n \setminus \{0\}$. For $\nu_i = 0$ the function $f_i$ may have discontinuity at the origin. Therefore, we need to develop an identifier which can deal with discontinuous models under considerations. Classical approximation theorems does not work in this case, since they usually assume (at least) continuity of the model on a compact.

## 3. NEURAL NETWORKS APPROXIMATION PROPERTY FOR HOMOGENEOUS SYSTEMS

The universal approximation property of NN has been used in many works, which says that any continuous function can be approximated arbitrarily closely on a compact set using superposition of nonlinear functions such as polynomials, radial basis functions and sigmoidal functions (Hornik et al., 1989), (Cybenko, 1989).

*Theorem 4.* (Haykin (1994)). *Let $\sigma(\cdot)$ be a bounded monotone-increasing continuous function and let $C(I_p^c)$ be the space of continuous functions defined in the unit hypercube of $p$ dimensions $I_p^c := [0,\ 1]^p$. Then, given any function $f \in C(I_p^c)$ and $\varepsilon > 0$, there exists an integer $N$ and sets of real constants $\alpha_i$, $b_i$, $\omega_{ij}$, where $i = 1, 2, ..., N$ and $j = 1, 2, ..., p$, such that the following representation:*

$$G(x) := \sum_{i=1}^{N} \alpha_i \sigma\left(\sum_{j=1}^{p} \omega_{ij} x_j + b_i\right), \tag{5}$$

*is an approximate realization of the function $f(\cdot)$, i.e.*

$$|G(x) - f(x)| < \varepsilon,\ \forall x \in I_p^c. \tag{6}$$

Notice that Theorem 4 is formulated for a two layers static NN structure, the proof of this theorem is based on the Stone–Weisstrass and the Kolmogorov approximation theorems, this property is used in DNN structures for the representation of dynamic systems.

*Remark 5.* In (Lewis et al., 1998) and (Igelnik and Pao, 1995), it is stated that the parameters $\omega_{ij}$, and $b_i$ in the structure presented in (5) can be selected randomly with a uniform distribution. Then, in the aforementioned works, it is shown that the universal approximation property holds by finding only the parameters $\alpha_i$.

The next result uses the universal approximation property of NN structures for the case of the dynamic nonlinear system (1) satisfying assumptions 2 and 3.

*Corollary 6. Let the system* (1) *satisfy the assumptions 1, 2 and 3. Then, for any $\varepsilon_i \in \mathbb{R}_+$ and for any matrix $A \in \mathbb{R}^{n\times n}$ there exist $W_i \in \mathbb{R}^{n\times N_i}$, $i = 0, 1, .., m$ such that:*

$$\|f(x,u) - F(x,u)\| \leq \varepsilon_0 \|x\|^{\nu_0} + \sum_{i=1}^{m} \varepsilon_i \|x\|^{\nu_i} |u_i|$$

$$\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m,$$

*where*

$$F(x,u) := \|x\|^{\nu_0}\left[\tfrac{Ax}{\|x\|} + W_0 \sigma_0\left(\tfrac{x}{\|x\|}\right)\right] + \sum_{i=1}^{m} \|x\|^{\nu_i} W_i \sigma_i\left(\tfrac{x}{\|x\|}\right) u_i \tag{7}$$

*and the elements of the vector fields $\sigma_i(\cdot)$ are sigmoidal activation functions:*

$$(\sigma_i(x))_j = \alpha_{ij}\left(1 + c_{ij}e^{-b_{ij}^\top x}\right)^{-1}, \tag{8}$$

*where $\alpha_{ij} \in \mathbb{R}_+$, $c_{ij} \in \mathbb{R}_+$ and $b_{ij} \in \mathbb{R}^n$ are some properly selected parameters with $i = 0, 1, ..., m$ and $j = 1, ..., N_i$.*

**Proof.** Obviously that under Assumption 2 the system (1) can be rewritten as:

$$\dot{x} = \|x\|^{\nu_0} f_0\left(\tfrac{x}{\|x\|}\right) + \sum_{i=1}^{m} \|x\|^{\nu_i} f_i\left(\tfrac{x}{\|x\|}\right) u_i, \tag{9}$$

i.e. the right-hand side of the system (1) is uniquely identified by its values on the unit sphere (2).

Let us consider the functions $\tilde{f}_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, 1, ..., m$ defined as follows

$$\tilde{f}_i(x) = f_i\left(\tfrac{x}{\|x\|}\right).$$

Obviously, $\tilde{f}_i$ is continuous on $\mathbb{R}^n$ due to continuity of $f_i$ on the unit sphere.

Applying Theorem 4 to each component of the vector $\tilde{f}_i$, $i = 1, 2, ...m$ and to $\tilde{f}_0\left(\tfrac{x}{\|x\|}\right) - A\tfrac{x}{\|x\|}$ on the unit sphere $S$, we obtain that $W_i\sigma_i\left(\tfrac{x}{\|x\|}\right)$ approximates $\tilde{f}_i(x) = f_i\left(\tfrac{x}{\|x\|}\right)$ with an error $\varepsilon_i$, and this error can be made arbitrary small by means of a proper selection of parameters $W_i$, $\alpha_{ij}$, $b_{ij}$ and $c_{ij}$. Finally, taking into account homogeneity of nonlinear functions $f_i, i = 0, ..., m$ we complete the proof.

In the view of Remark 5 below we assume that appropriate parameters $\alpha_{ij}, b_{ij}$ and $c_{ij}$, are somehow selected, and we just need to find matrices $W_i$ in order to identify the model.

## 4. IDENTIFICATION OF AFFINE HOMOGENEOUS CONTROL SYSTEMS

### *4.1 The case of known control gains*

Let us consider initially the case when the nonlinear maps $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, ..., m$ associated with the inputs are known and *we need to identify only the vector field $f_0$*.

In the most straightforward case, system (1) is represented with an exact NN matching ($\varepsilon_0 = 0$). The following modeling considers that for a selected Hurwitz matrix $A \in \mathbb{R}^{n\times n}$, there exists a weight matrix $W_0^* \in \mathbb{R}^{n\times N_0}$ such that $f_0$ can be exactly presented by the NN structure described in Corollary 6. Notice that

$$W_0^*\sigma_0\left(\tfrac{x}{\|x\|}\right) := \Sigma_0\left(\tfrac{x}{\|x\|}\right) w_0^*, \tag{10}$$

where

$$\Sigma_0(z) = I_n \otimes \sigma_0^\top(z) \in \mathbb{R}^{n\times nN_0}, \quad z \in \mathbb{R}^n,$$

$I_n \in \mathbb{R}^{n\times n}$ is the identity matrix and $w_0^* = \mathrm{vec}\left((W_0^*)^\top\right) \in \mathbb{R}^{nN_0}$.

The identification problem can be seen as finding $w_0^*$ by using some learning law such that $x$ can be reproduced by $\hat{x}$, where $\hat{x} \in \mathbb{R}^n$ represents the state vector of the DNN identifier (whose equations are given below in (11)).

The following theorem introduces the first result for the convergence of the identification error.

*Theorem 7. Let assumptions 1, 2 and 3 be satisfied and the homogeneous vector field $f_0$ admits the exact representation* (7) *and the DNN identifier is defined as follows:*

$$\tfrac{d}{dt}\hat{x} = \|x\|^{\nu_0}\left[\tfrac{A\hat{x}}{\|x\|} + \Sigma_0\left(\tfrac{x}{\|x\|}\right) w_0 + \Omega K \Omega^\top e\right] + \sum_{i=1}^{m} f_i(x) u_i, \tag{11}$$

*where $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix, $e := x - \hat{x}$ is the identification error, $w_0 \in \mathbb{R}^{nN_0}$ is the vector of the weights to be adjusted as follows*

$$\frac{d}{dt}w_0 = -\|x\|^{\nu_0} K \Omega^\top e \tag{12}$$

*and $\Omega \in \mathbb{R}^{n\times nN_0}$ is an auxiliary variable satisfying*

$$\frac{d}{dt}\Omega = \|x\|^{\nu_0}\left(A\tfrac{\Omega}{\|x\|} - \Sigma_0\left(\tfrac{x}{\|x\|}\right)\right). \tag{13}$$

*If $K \in \mathbb{R}^{nN_0\times nN_0}$ is a positive definite matrix and*

$$\int_t^{t+\ell} \Omega^\top(\tau)\Omega(\tau)d\tau \geq \upsilon I_{nN_0}, \quad \forall t \in \mathbb{R}_+, \tag{14}$$

*for some $\ell > 0$ and $\upsilon > 0$. then, the vector of weights $w_0$ converge to $w_0^*$ asymptotically and the identification process is asymptotically consistent, i.e.*

$$\lim_{t\to\infty} e(t) = 0.$$

**Proof.** To prove the stability of the DNN as well as the stability of the identification error $e$, let us define the following auxiliary equation:

$$\delta = e + \Omega\tilde{\omega}, \tag{15}$$

where $\tilde{\omega} := w_0^* - w_0$. Hence, the dynamics of the auxiliary equation is:

$$\dot{\delta} = \dot{e} + \frac{d\Omega}{dt}\tilde{\omega} + \Omega\frac{d\tilde{\omega}}{dt}, \tag{16}$$

the dynamics of the identification error is:

$$\dot{e} = \|x\|^{\nu_0}\left[\tfrac{Ae}{\|x\|} + \Sigma_0\left(\tfrac{x}{\|x\|}\right)\tilde{\omega} - \Omega K \Omega^\top e\right], \tag{17}$$

the time derivative $\frac{d\tilde{\omega}}{dt} = -\frac{dw_0}{dt}$, by substituting (12), (13) and (17) in (16), we have:

$$\begin{aligned}\dot{\delta} = \|x\|^{\nu_0}\left[A\tfrac{e}{\|x\|} + \Sigma_0\left(\tfrac{x}{\|x\|}\right)\tilde{\omega} - \Omega K \Omega^\top e\right] + \\ \|x\|^{\nu_0}\left(A\tfrac{\Omega}{\|x\|} - \Sigma_0\left(\tfrac{x}{\|x\|}\right)\right)\tilde{\omega} + \|x\|^{\nu_0}\Omega K \Omega^\top e = \\ \|x\|^{\nu_0}\left(A\tfrac{e}{\|x\|} + A\tfrac{\Omega}{\|x\|}\tilde{\omega}\right) = \|x\|^{\nu_0}\tfrac{A}{\|x\|}\delta.\end{aligned} \tag{18}$$

Since the matrix $A$ is Hurwitz and $\|x\| \neq 0$, then the condition (14) implies $\delta \to 0$ and $e \to -\Omega\tilde{\omega}$ as $t \to +\infty$. Since the matrix $A$ is Hurwitz and the state $x(t)$ is bounded due to Assumption 3, it is straightforward to observe that the variable $\Omega(t)$ is also bounded.

Using the selected learning law (12) and the fact on the stability of the dynamic auxiliary equation (16), we obtain:

$$\frac{d\tilde{\omega}(t)}{dt} = \|x\|^{\nu_0} K \Omega^\top(\Omega\tilde{\omega} - \delta), \tag{19}$$

hence, since the condition of persistence excitation (14) is satisfied, the latter system is input-to-state stable with respect to the input $\delta$ (see e.g. Efimov and Fradkov (2015)). Therefore, for asymptotically converging signal $\delta$, the asymptotic convergence of $\tilde{\omega}$ (and $e$, respectively since both $\delta$ and $\tilde{\omega}$ are converging) to zero can be established.

Notice that the DNN identifier (11) is not regular (Poznyak et al., 2001), (Chairez, 2017) because it has a direct injection of the identification error $e$.

*4.2 The case of unknown control gains*

The previous design considers the nonlinear part associated to the input as known. In this section we design a DNN identifier for the considered system assuming that functions $f_i$ are unknown but representation (7) admits the exact match ($\varepsilon_0 = 0$ and $\varepsilon_i = 0$).

Similarly to (10) we introduce the vectors $w_i^*$ and the matrix-valued function $\Sigma_i$ such that

$$W_i^* \sigma_i\left(\tfrac{x}{\|x\|}\right) := \Sigma_i\left(\tfrac{x}{\|x\|}\right) w_i^*.$$

*Theorem 8. Let the system* (1) *admits an exact representation of the form* (7)*, assumptions 1-3 be satisfied and the control input u be selected as follows:*

$$u_i(t) = \frac{\tilde{u}_i(t)}{\|x(t)\|^{\nu_i - \nu_0}}, \quad i = 1, \ldots, m, \tag{20}$$

*where $\tilde{u}_i$ are continuous uniformly bounded functions and the DNN identifier is defined as follows*

$$\tfrac{d\hat{x}}{dt} = \|x\|^{\nu_0 - 1} A\hat{x} + \sum_{i=0}^{m} \|x\|^{\nu_0}\left[\Sigma_i\left(\tfrac{x}{\|x\|}\right) w_i \tilde{u}_i + \Omega_i K_i \Omega_i^\top e\right], \tag{21}$$

*where $u_0 \equiv 1$, $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix and $e := x - \hat{x}$ is the identification error, $w_i \in \mathbb{R}^{nN_i}$ are the vectors of the weights to be adjusted as follows:*

$$\frac{d}{dt} w_i = -\|x\|^{\nu_0} K_i \Omega_i^\top e \tag{22}$$

*and $\Omega_i \in \mathbb{R}^{n\times nN_i}$ are auxiliary variables satisfying:*

$$\frac{d}{dt}\Omega_i = \|x\|^{\nu_0 - 1} A\Omega_i - \tilde{u}_i \|x\|^{\nu_0} \Sigma_i\left(\tfrac{x}{\|x\|}\right), \tag{23}$$

*If $K_i \in \mathbb{R}^{nN_i \times nN_i}, i = 0, 1, \ldots, m$ are positive definite matrices and:*

$$\int_t^{t+\ell_W} H^\top(\tau) H(\tau) d\tau \geq \upsilon_W I_{\sum_{i=0}^m N_i} \tag{24}$$

*for all $t \in \mathbb{R}_+$, some $\ell_W > 0$, $\upsilon_W > 0$, where $H \in \mathbb{R}^{n\times \sum_{i=0}^m N_i}$ is a block matrix $H = \{\Omega_i\}$. Then, the vectors of weights $w_i$ converge to $w_i^*$ asymptotically and the identification process is asymptotically consistent, i.e.*

$$\lim_{t\to\infty} e(t) = 0.$$

**Proof.** To prove the stability of the DNN as well as the stability of the identification error $e$, let us define the following auxiliary equation:

$$\delta = e + \sum_{i=0}^{m} \Omega_i \tilde{\omega}_i, \tag{25}$$

where $\tilde{\omega}_i := w_i^* - w_i$. Hence, the dynamics of the auxiliary variable is:

$$\begin{aligned}
\dot{\delta} &= \dot{e} + \sum_{i=0}^{m} \tfrac{d\Omega_i}{dt}\tilde{\omega}_i + \Omega_i \tfrac{d\tilde{\omega}_i}{dt} \\
&= \|x\|^{\nu_0 - 1} Ae + \sum_{i=0}^{m} \|x\|^{\nu_0}\left[\Sigma_i\left(\tfrac{x}{\|x\|}\right)\tilde{\omega}\tilde{u}_i - \Omega_i K_i \Omega_i^\top e\right] \\
&+ \sum_{i=0}^{m}\left(\|x\|^{\nu_0 - 1} A\Omega_i - \tilde{u}_i \|x\|^{\nu_0}\Sigma_i\left(\tfrac{x}{\|x\|}\right)\right)\tilde{\omega}_i + \|x\|^{\nu_i}\Omega_i K \Omega_i^\top e \\
&= \|x\|^{\nu_0 - 1}\left(Ae + A\textstyle\sum_{i=0}^m \Omega_i\tilde{\omega}_i\right) = \|x\|^{\nu_0 - 1} A\delta,
\end{aligned}$$

Since the matrix $A$ is Hurwitz and $\|x\| \neq 0$, then the condition (24) implies $\delta \to 0$ and $e \to -\sum_{i=0}^m \Omega_i \tilde{\omega}_i$ as $t \to +\infty$. The variables $\Omega_i(t)$ are bounded due to the structure of (23), Hurwitz property of the matrix $A$, boundedness and separation from zero of $u_i(t)$ and $\|x(t)\|$.

Using the selected learning law (22) and the fact on the stability of the dynamic auxiliary equation (25), we obtain:

$$\frac{d\tilde{\omega}_i(t)}{dt} = \|x\|^{\nu_i} K_i \Omega_i^\top \left(\sum_{j=0}^{m} \Omega_j \tilde{\omega}_j - \delta\right), \tag{26}$$

hence, since the condition of persistence excitation (24) is satisfied, the latter system is input-to-state stable with respect to the input $\delta$ (see e.g. Efimov and Fradkov (2015)). Therefore, for asymptotically converging signal $\delta$, the asymptotic convergence of $\tilde{\omega}_i$ (and $e$, respectively) to zero can be established.

## 5. NUMERICAL RESULTS

Let us consider the three tank system depicted in figure 1.

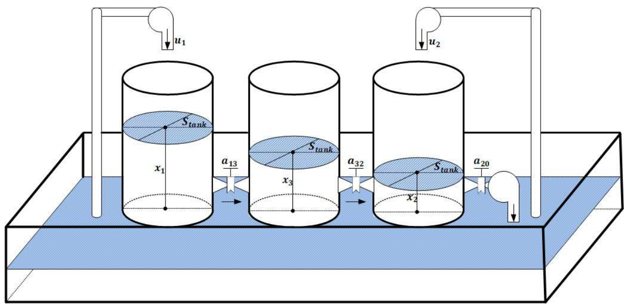


Fig. 1. Three tank system

This nonlinear system can be modeled as in (Join et al., 2005; Seydou et al., 2013; Zimenko et al., 2017) by:

$$\begin{aligned}
\dot{x}_1 &= \tfrac{1}{S_{tank}}\left[-a_{13}\lceil x_1 - x_3 \rfloor^{0.5} + u_1\right], \\
\dot{x}_2 &= \tfrac{1}{S_{tank}}\left[a_{32}\lceil x_3 - x_2 \rfloor^{0.5} - a_{20}\lceil x_2 \rfloor^{0.5} + u_2\right], \\
\dot{x}_3 &= \tfrac{1}{S_{tank}}\left[a_{13}\lceil x_1 - x_3 \rfloor^{0.5} - a_{32}\lceil x_3 - x_2 \rfloor^{0.5}\right],
\end{aligned} \tag{27}$$

where $x_1$, $x_2$ and $x_3$ represent the liquid level of each tank respectively, $S_{tank}$ is the diameter of the three tanks, the input flows $u_1$ and $u_2$ are the control signals and the constant parameters $a_{13}$, $a_{32}$ and $a_{20}$ are coefficients related with the outflow rate according to Torricelli's rule.

Checking assumptions 2 and 3 for the system (27), it is straightforward to check that the system is of homogeneity degree $\nu = -0.5$. In table 1, the values used to simulate the three thank system are presented, this values are not used on the identifier.

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $S_{tank}$ | Tank diameter | $1m$ |
| $a_{13}$ | Outflow rate coefficient | $3\ m^{\frac{3}{2}}/s^2$ |
| $a_{32}$ | Outflow rate coefficient | $2\ m^{\frac{3}{2}}/s^2$ |
| $a_{20}$ | Outflow rate coefficient | $1\ m^{\frac{3}{2}}/s^2$ |
| $x_0$ | Initial conditions | $[3,\ 1,\ 2]^\top$ |
| $\lambda$ | Real positive value | 0.4 |

Table 1. Parameters for the simulation of the three tank system

The numerical simulations for the identifiers were made in Simulink Matlab® by using the Runge Kutta integration method with a step of 1 $ms$.

For the first identifier, let us consider the vector of activation functions $\sigma_0(\cdot)$ in (8) with $N_0 = 3$ and the constant parameters $\alpha_{0,1} = 2.7$, $\alpha_{0,2} = 1.8$, $\alpha_{0,3} = 3.6$, $c_{0,1} = 8$, $c_{0,2} = 16$, $c_{0,3} = 8$, the constant vectors:

$$b_{0,1} = \begin{bmatrix} 0.01 \\ 0.02 \\ 0.03 \end{bmatrix},\ b_{0,2} = \begin{bmatrix} 0.01 \\ 0.04 \\ 0.01 \end{bmatrix},\ b_{0,3} = \begin{bmatrix} 0.04 \\ 0.01 \\ 0.06 \end{bmatrix}, \tag{28}$$

the matrix $A$ and the functions vectors $f_i$ are :

$$A := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -1 \end{bmatrix};\ f_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\ f_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$f_0(x) = \frac{1}{S_{tank}} \begin{pmatrix} -a_{13}\lceil x_1 - x_3 \rfloor^{0.5} \\ a_{32}\lceil x_3 - x_2 \rfloor^{0.5} - a_{20}\lceil x_2 \rfloor^{0.5} \\ a_{13}\lceil x_1 - x_3 \rfloor^{0.5} - a_{32}\lceil x_3 - x_2 \rfloor^{0.5} \end{pmatrix}, x = (x_1, x_2, x_3)^\top.$$

The initial conditions for the adjustment law are $w_0(0) = [1\ 2\ 1\ 1\ 3\ 2\ 1\ 2\ 3]^\top$ and the gain matrix is selected as $K := I_3 \otimes \tilde{K}$, where the matrix $\tilde{K}$ is defined as:

$$\tilde{K} := \begin{bmatrix} 100 & -20 & 30 \\ -20 & 30 & 10 \\ 30 & 10 & 100 \end{bmatrix}.$$

For the second identifier with an unknown nonlinear term associated to the input was used the same integration method and matrix $A$. The activation functions were selected as in (8) with all $N_i = 3$. The constant parameters $\alpha_{0,j}$, $c_{0,j}$ and the vectors $b_{0,j}$ are the same as the used in the first identifier. The constant parameters for $\sigma_i(\cdot)$ were selected as $\alpha_{i,j} = \alpha_{0,j}$ and $c_{i,j} = c_{0,j}$ and the vectors $b_{i,j} = b_{0,j}$.

The initial conditions for the adjustment law were selected as $w_0(0)$ is the same as the used for the first identifier and $w_1(0) = w_0(0)$ and $w_2(0) = w_0(0)$ and the gain matrices $K_i := I_3 \otimes \tilde{K}$,

Figures 2, 3, 4 and 5 show the obtained results. The first identifier (– blue) uses a known gain of the input. Its state estimate converges quickly to the state of the uncertain system (– black). The convergence rate of the second identifier (··· red) is much slower but it does not assume that functions $f_1$ and $f_2$ are known.

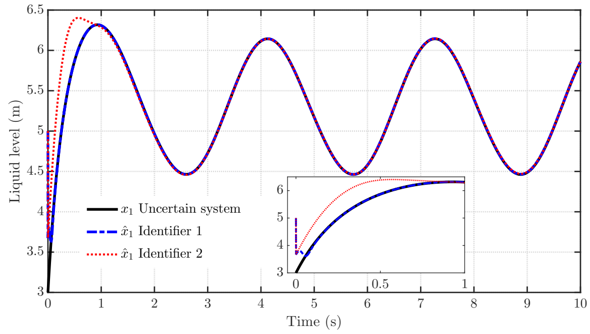


Fig. 2. Identification result for the first state ($x_1$)

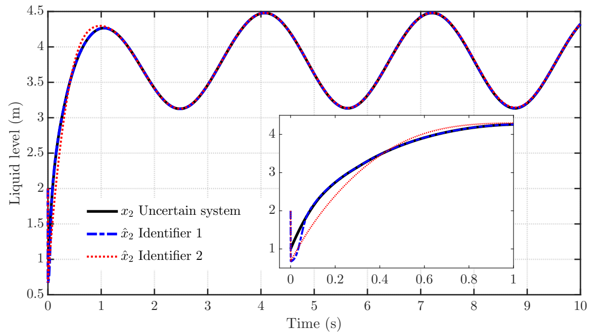


Fig. 3. Identification result for the second state ($x_2$)

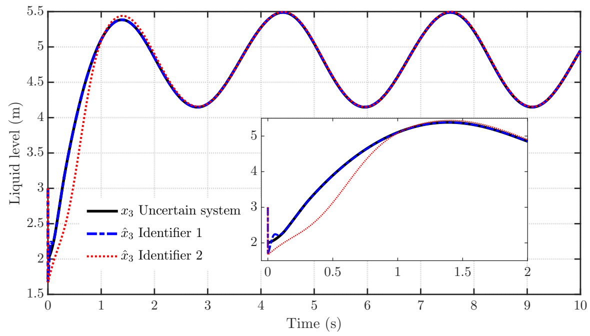


Fig. 4. Identification result for the third state ($x_3$)

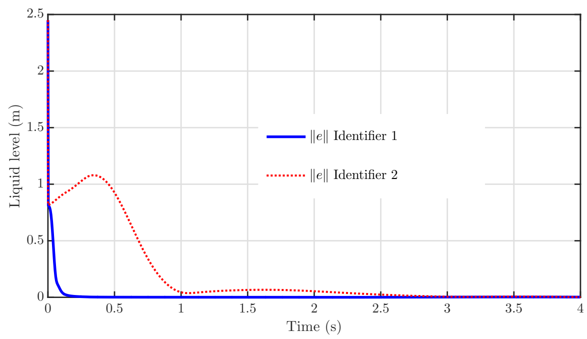


Fig. 5. Norm of the identification error

## 6. CONCLUSIONS

In this paper, DNN identifiers for affine standard homogeneous (possibly discontinuous) uncertain control system are developed. Their convergence is proven theoretically and their precision confirmed by numerical simulation for homogeneous model of three tank system. The main assumption required for the design of a homogeneous DNN identifier is knowledge of the homogeneity degree. The problem of identification of homogeneity degree is planned to be studied in a recent future.

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